

TENSOR, SOBOLEV, MULTIPLICATIVE AND CONVOLUTION OPERATORS IN THE BIDE - SIDE GRAND LEBESQUE SPACES

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ABSTRACT. In this paper we study the multiplicative, tensor, Sobolev's and convolution inequalities in certain Banach spaces, the so-called Bide - Side Grand Lebesgue Spaces, and give examples to show their sharpness.

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1. INTRODUCTION

Let (X, Σ, μ) be a measurable space with non - trivial measure μ . We suppose the measure to be diffuse, that is, for all $A \in \Sigma$ such that $\mu(A) \in (0, \infty)$ and there exists $B \subset A$ such that $\mu(B) = \mu(A)/2$. We also suppose that the measure is σ -finite: there is a sequence $E(n) \in \Sigma$ such that $\mu(E(n)) < \infty$ and $\cup_{n=1}^{\infty} E(n) = X$.

For a and b constants, $1 \leq a < b \leq \infty$, let $\psi = \psi(p)$, $p \in (a, b)$ be a continuous log-convex positive function such that $\psi(a+0)$ and $\psi(b-0)$ exist, with $\max(\psi(a+0), \psi(b-0)) = \infty$ and $\min(\psi(a+0), \psi(b-0)) > 0$.

The Bide-Side Grand Lebesgue Space $BSGLS(\psi; a, b) = G_X(\psi; a, b) = G(\psi; a, b) = G(\psi)$ consists, by the well-known definition, of all measurable functions $h : X \rightarrow \mathbb{R}$ with finite norm

$$\|h\|_{G(\psi)} \stackrel{def}{=} \sup_{p \in (a, b)} |h|_p / \psi(p), \quad |h|_p = \left[\int_X |h(x)|^p \mu(dx) \right]^{1/p}.$$

These spaces are intensively studied, in particular, their associate spaces, fundamental functions $\phi(\delta; G(\psi; a, b))$, Fourier and singular operators, condition for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.; see, e.g., [1, 3, 4, 5, 6, 7, 8, 10, 12, 15]. They are also Banach and moreover rearrangement invariant (r.i.) spaces.

The BSGLS norm, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [4, 5, 6, 7], probability in Banach spaces [13], in the modern non-parametrical statistics, for example, in the so-called regression problem [14, 15]. The latter reads as follows. Given the observation $\{\xi(i)\}$, $i = 1, 2, 3, \dots, n$, with $n \rightarrow \infty$, of the view

$$\xi(i) = g(z(i)) + \epsilon(i), \quad i = 1, 2, \dots,$$

where $g(\cdot)$ is an unknown estimated function, $\{\epsilon(i)\}$ are the errors of measurements that may be independent random variables or martingale differences, $\{z(i)\}$ is a dense set in the metric space (Z, ρ) with Borel measure ν .

Let $\{\phi_k(z)\}$ be a complete orthonormal sequence of functions, for example, the classical trigonometric sequence, Legendre or Hermite polynomials, etc. Put

$$c_k(n) = n^{-1} \sum_{i=1}^n \phi_k(z(i)) \quad , \quad \tau(N) = \tau(N, n) = \sum_{k=N+1}^{2N} (c_k(n))^2,$$

$$M = \arg \min_{N \in [1, n/3]} \tau(N) \quad , \quad f_n(z) = \sum_{k=1}^M c_k(n) \phi_k(z).$$

To study the confidence region for estimating function f in the $L(p)$ norm, written $|f_n - f|_p$, exponential bounds for the tail of the distribution of polynomial martingales are used being derived via the BSGLS spaces.

Let $a \geq 1$, $b \in (a, \infty]$, and let $\psi = \psi(p)$ be a positive continuous function on the *open* interval (a, b) such that there exists a measurable function $f : X \rightarrow \mathbb{R}$ for which

$$(1) \quad f(\cdot) \in \cap_{p \in (a, b)} L_p, \quad \psi(p) = |f|_p, \quad p \in (a, b).$$

We say that the equality (1) and the function $f(\cdot)$ from (1) is the *representation* of the function ψ .

We denote the set of all these functions by Ψ : $\Psi = \Psi(a, b) = \{\psi(\cdot)\}$. For complete description of these functions see, for example, ([14, p.p. 21 - 27], [15]).

Remark 1. Observe that if $\psi_1 \in \Psi(a_1, b_1)$ and $\psi_2 \in \Psi(a_2, b_2)$, with $(a_1, b_1) \cap (a_2, b_2) = (a_3, b_3) \neq \emptyset$, then $\psi_1 \psi_2 \in \Psi(a_3, b_3)$. Indeed, if $\psi_1(p) = |f_1|_p$ and $\psi_2(p) = |f_2|_p$, and the functions f_1 and f_2 are independent in the probabilistic sense, that is, for all Borel sets A, B on the real axis \mathbb{R}

$$\mu\{x : f_1(x) \in A, f_2(x) \in B\} = \mu\{x : f_1(x) \in A\} \mu\{x : f_2(x) \in B\},$$

then $\psi_1(p) \psi_2(p) = |f_1 f_2|_p$.

We note that the $G(\psi)$ spaces are also interpolation spaces (the so-called Σ -spaces), see [1, 3, 4, 5, 6, 7, 8, 12, 15], etc. However, we hope that our direct representation of these spaces is of certain convenience in both theory and applications.

The $G(\psi)$ spaces with $\mu(X) = 1$ appeared in [11]; it was proved that in this case each $G(\psi)$ space coincides with certain exponential Orlicz space, up to norm equivalence.

The main goal of this paper is to prove new (and extend known) results on the Boyd's indices, tensor, Sobolev embedding, multiplicative and convolution operators in BSGLS spaces.

The paper is organized as follows. In the next section we start with an exemplary case just to give a feeling of what happens and then present the main results on the so-called Boyd index of BSGLS – spaces, on tensors and multiplicative inequalities, and the Sobolev embedding theorem and convolution inequalities, each of these in a separate subsection. Further, the section follows where we prove the statements, and in the last section we discuss the sharpness of the obtained results.

2. RESULTS

To get a flavor of the setting we work with, let, for instance, $X = \mathbb{R}^n$, σ be a constant, and $\mu = \mu_\sigma$ be the measure on Borel subsets of X with density $d\mu_\sigma/dx = |x|^\sigma$. As usual, $x = (x_1, x_2, \dots, x_n) \in X$ so that $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$. let $L = L(z)$, $z \in (0, \infty)$ be a slowly varying as $z \rightarrow \infty$ positive continuous function, $I(A) = I(A, x) = 1$ for $x \in A$ and $I(A, x) = 0$ otherwise. Let

$$\begin{aligned} f_L(x) &= f(x; L, a, \alpha) = I(|x| > 1) |x|^{-1/a} [\log(|x|)]^\alpha L(\log |x|), \\ g_L(x) &= g(x; L, b, \beta) = I(|x| < 1) |x|^{-1/b} [|\log(|x|)|]^\beta L(|\log |x||), \\ A = A(a, n, \sigma) &= a(n + \sigma) \geq 1, \quad B = B(b, n, \sigma) = b(n + \sigma) \in (A, \infty), \\ \gamma &= \alpha + 1/A, \quad \delta = \beta + 1/B, \quad p \in (A, B), \end{aligned}$$

and

$$\psi_L(p) = \psi_L(p; A, B; \gamma, \delta) \stackrel{\text{def}}{=} (p - A)^{-\gamma} (B - p)^{-\delta} \max(L(A/(p - A)), L(B/(B - p))).$$

The function $h_L(x) = h_L(x; a, b; \alpha, \beta) = f_L(x) + g_L(x)$ belongs to the space $G(\psi_L)$:

$$h_L(\cdot) \in G(A, B; \gamma, \delta) \stackrel{\text{def}}{=} G(\psi_L(\cdot)),$$

and this inclusion is exact in the sense that for $p \in (A, B)$ there holds $|h_L|_p \asymp \psi_L(p)$, where here and in what follows for $p \in (A, B)$ the relation $g(p) \asymp h(p)$ denotes

$$0 < \inf_{p \in (A, B)} f(p)/g(p) \leq \sup_{p \in (A, B)} f(p)/g(p) < \infty.$$

At the endpoints we need more in the case when $f(p) \rightarrow \infty$. This may occur when either $p \rightarrow A+$ or $p \rightarrow B-$ or in both cases. In detail, this means that in the case when $\psi(A + 0) = \infty$ while $\psi(B - 0) < \infty$ there holds

$$\lim_{p \rightarrow A+0} \psi(p)/\nu(p) = 1;$$

in the case when $\psi(B - 0) = \infty$ while $\psi(A + 0) < \infty$ there holds

$$\lim_{p \rightarrow B-0} \psi(p)/\nu(p) = 1;$$

and when in both cases $\psi(A + 0) = \psi(B - 0) = \infty$ there holds

$$\lim_{p \rightarrow A+0} \psi(p)/\nu(p) = \lim_{p \rightarrow B-0} \psi(p)/\nu(p) = 1.$$

Denoting now $\omega(n) = \pi^{n/2}/\Gamma(n/2 + 1)$ and $\Omega(n) = n\omega(n) = 2\pi^{n/2}/\Gamma(n/2)$, we let

$$R = R(\sigma, n) = [(\sigma + n)/\Omega(n)]^{1/(\sigma + n)}, \quad \sigma + n > 0,$$

and let $h = h(|x|)$ be a non-negative measurable function vanishing for $|x| \geq R(\sigma, n)$. For $u \geq e^2$ let

$$\mu_\sigma\{x : h(|x|) > u\} = \min(1, \exp(-W(\log u))),$$

where $W = W(z)$ is a twice differentiable strictly convex for $z \in [2, \infty)$ and strictly increasing function. Denoting by

$$W^*(p) = \sup_{z>2} (pz - W(z))$$

the Young - Fenchel transform of the function $W(\cdot)$, we define the function

$$\psi(p) = \exp(W^*(p)/p).$$

It follows from the theory of Orlicz spaces ([14, p.p. 22 - 27]) that if for $p \in [a, \infty)$ we have $|h|_p \asymp \psi(p)$, then $h(\cdot) \in G(\psi; a, \infty)$ and $G(\psi; 1, \infty)$ coincides with some exponential Orlicz space.

We will restrict ourselves to the case $p \in (a(n+\sigma), b(n+\sigma))$, $a(n+\sigma) \geq 1$, $b(n+\sigma) < \infty$, with $p \rightarrow a(n+\sigma) + 0$. Denoting $A = a(n+\sigma)$, let

$$f(x) = f_L(x) = I(|x| > 1) |x|^{-1/a} (\log |x|)^\gamma L(\log |x|).$$

Using then multidimensional polar coordinates and well-known properties of slowly varying functions ([17, Ch.1, Sect.1.4 - 1.5]), we obtain

$$\begin{aligned} \Omega(n) \|f\|_p^p &= \int_1^\infty r^{-p/a+n+\sigma-1} (\log r)^\gamma L^p(\log r) dr \\ &= (p/a - n - \sigma)^{-\gamma p-1} \int_0^\infty e^{-z} z^{\gamma p} L^p(az/(p-A)) dz \\ &\sim (p/a - n - \sigma)^{-\gamma p-1} L^p(a/(p-A)) \int_0^\infty e^{-z} z^{\gamma p} dz \\ &= (p/a - n - \sigma)^{-\gamma p-1} L^p(a/(p-A)) \Gamma(1 + \gamma p). \end{aligned}$$

Thus, we have for $p \in (A, B)$

$$|f_L|_p \asymp (p-A)^{-\gamma-1/A} L(a/(p-A)).$$

2.1. Indices. In this section we give an expression for the so-called Boyd's (and other) indices of $G(\psi, a, b)$ spaces in the case of $X = [0, \infty)$ with usual Lebesgue measure. These indices play very important role in the theory of interpolation of operators, in Fourier Analysis on r.i. spaces, etc. (see, e.g., [1, p.p. 22 - 31, 192 - 204]).

We recall the definitions. Given the family of (linear) operators $\{\sigma_s\}$ acting from some r.i. space G to G by the following definition:

$$\sigma_s f(x) = f(x/s), \quad s > 0, \quad \|\sigma_s\| = \|\sigma_s\|_{G \rightarrow G}.$$

We have for arbitrary r.i. space G on the set X by the classical definition ([1], chapter 2, [11], chapter 2)

$$\begin{aligned}\gamma_1(G) &= \lim_{s \rightarrow 0+} \log \|\sigma_s\| / \log s; & \gamma_2(G) &= \lim_{s \rightarrow \infty} \log \|\sigma_s\| / \log s; \\ \gamma^{(2)}(G) &= \limsup_{s \rightarrow 0+} \phi(G; 2s) / \phi(G; s); & \gamma^{(1)}(G) &= \liminf_{s \rightarrow 0+} \phi(G; 2s) / \phi(G; s).\end{aligned}$$

Theorem 2. *There holds*

$$\gamma_1(G(\psi; a, b)) = 1/b, \quad \gamma_2(G(\psi; a, b)) = 1/a, \quad \psi \in \Psi;$$

and if the space X is arbitrary and the measure μ is diffuse

$$\gamma^{(1)}(G(\psi; a, b)) = \gamma^{(2)}(G(\psi; a, b)) = 2^{1/b}.$$

In a more general case of $X = R^n$ with $\mu = \mu_\sigma$ and $\sigma \geq 0$, we analogously have

$$\gamma_1(G(\psi; a, b)) = (n + \sigma)/b, \quad \gamma_2(G(\psi; a, b)) = (n + \sigma)/a, \quad \psi \in \Psi.$$

2.2. Tensor and multiplicative inequalities. Let (X, Σ_1, μ) and (Y, Σ_2, ν) be two measurable spaces with σ -finite measures μ and ν respectively. Let $f = f(x) \in G_X(\psi_1; a_1, b_1)$ and $g = g(y) \in G_Y(\psi_2; a_2, b_2)$, where $x \in X$, $y \in Y$, $\psi_1, \psi_2 \in \Psi$, and let $a = \max(a_1, a_2) < \min(b_1, b_2) = b$. We set $\psi(p) \stackrel{\text{def}}{=} \psi_1(p) \psi_2(p)$ for $p \in (a, b)$.

Let us consider the so-called *tensor product* of $f, g : z(x, y) \stackrel{\text{def}}{=} f(x) g(y)$. Since both functions f and g are independent of the space $(X \times Y, \Sigma_1 \times \Sigma_2, \zeta)$, with $\zeta = \mu \times \nu$, we have:

Lemma 1. *The following tensor inequality holds*

$$\|z\|_{G(\psi; a, b)} \leq \|f\|_{G(\psi_1; a_1, b_1)} \|g\|_{G(\psi_2; a_2, b_2)}.$$

This inequality is sharp, for example, when $\psi(p) = |f|_p$ and $\psi_2(p) = |g|_p$.

We consider now the so-called *multiplicative* inequality. Let $f \in G(\psi_1; a_1, b_1)$ and $g \in G(\psi_2; a_2, b_2)$, with $a_1, a_2 \geq 1$ and $1/b_1 + 1/b_2 > 1$. We denote

$$A_1 = \max(1, a_1 a_2 / (a_1 + a_2)), \quad B_1 = b_1 b_2 / (b_1 + b_2),$$

and

$$\psi_3(r) = \inf\{\psi_1(pr) \psi_2(qr); p, q : p, q > 1, 1/p + 1/q = 1\},$$

with $r \in (A_1, B_1)$.

Theorem 3. *There holds*

$$(2) \quad \|f g\|_{G(\psi_3; A_1, B_1)} \leq \|f\|_{G(\psi_1; a_1, b_1)} \|g\|_{G(\psi_2; a_2, b_2)}.$$

We mention that the sharpness of (3) up to multiplicative constant can be seen from letting $f = f_L \in G(\psi_1)$ and $g = g_L \in G(\psi_2)$ with $\psi_1 = \psi_L(p; a_1, b_1; \alpha_1, \beta_1)$ and $\psi_2 = \psi_L(p, a_2, b_2; \alpha_2, \beta_2)$. Namely, in the considered case

$$(3) \quad \|f_L g_L\|_{G(\psi_3; A_1, B_1)} \geq C(L, a_1, a_2, b_1, b_2) \|f_L\|_{G(\psi_1; a_1, b_1)} \|g_L\|_{G(\psi_2; a_2, b_2)}.$$

We note, in addition, that if $f \in G(\psi; a, b)$ and for $\gamma = \text{const} \in [a, b]$ we have $\psi_\gamma(p) = \psi^\gamma(\gamma p)$, then $g(x) = f^\gamma(x) \in G(\psi_\gamma; a/\gamma, b/\gamma)$ and

$$\|f^\gamma\|_{G(\psi_\gamma)} = \|f\|_{G(\psi)}^\gamma.$$

2.3. Sobolev embedding and convolution operators. Let X be a convex domain in $R^n, n \geq 2$, with smooth boundary, B be a projection operator on the m -dimensional smooth convex sub-manifold Y of X , $m \leq n$, endowed with the corresponding surface measure, and let $\psi = \psi(p; a, b)$, $1 \leq a < b < n$. We denote $A_2 = \max(1, am/(n-a))$, $B_2 = bm/(n-b)$, $\nu(q) = q^{1-1/n}\psi(qn/(q+m))$, with $q \in (A_2, B_2)$, and $u = u(x) \in C_1^0(X)$, i.e. $u(\cdot)$ is continuous differentiable and $\lim_{|x| \rightarrow \infty} u(x) = 0$.

Theorem 4. *The following Sobolev type inequality holds*

$$(4) \quad \|Bu\|_{G(\nu; A_2, B_2)} \leq C(X, Y; \psi) \|\text{grad } u\|_{G(\psi; a, b)}.$$

This result is supplied with the following interesting assertion, opposite to that for the classical L_p spaces.

Theorem 5. *In Theorem 4 the corresponding embedding Sobolev operator is not compact.*

We now consider the (bilinear) generalized convolution operator of the form

$$v(x) = (f * g)(x) = \int_X g(xy^{-1})f(y) \mu(dy),$$

where X is an unimodular Lie's group, μ is its Haar's measure. The unimodularity means, in particular, that μ is bi-side invariant. For the commutative group X , with standard notation $y^{-1} = -y$, $xy^{-1} = x - y$, this definition coincides with the classical definition of convolution.

Let $f \in G(\psi_1; a_1, b_1)$ and $g \in G(\psi_2; a_2, b_2)$ provided $1/a_1 + 1/a_2 > 1$ and $1/b_1 + 1/b_2 > 1$. We denote

$$A_3 = a_1 a_2 / (a_1 + a_2 - a_1 a_2), \quad B_3 = b_1 b_2 / (b_1 + b_2 - b_1 b_2),$$

and for the values $r \in (A_3, B_3)$ we define

$$\tau(r) = \inf\{\psi_1(p) \psi_2(q); p, q : p, q > 1, 1/p + 1/q = 1 + 1/r\}.$$

Theorem 6. *There holds*

$$(5) \quad \|f * g\|_{G(\tau; A_3, B_3)} \leq \|f\|_{G(\psi_1; a_1, b_1)} \|g\|_{G(\psi_2; a_2, b_2)}.$$

3. PROOFS

Proof of Theorem 2.

The last assertion of the theorem follows from the explicit expression for the fundamental function $\phi(\delta, G(\psi; a, b))$, with $\delta \in (0, \infty)$:

$$\phi(\delta; G) = \sup\{\|I(A)\|_G, A \in \Sigma, \mu(A) \leq \delta\};$$

see [15]. The first assertion follows immediately from the identity

$$(6) \quad \|\sigma_s\| = \max(s^{1/a}, s^{1/b}), \quad s > 0, \psi \in \Psi,$$

where

$$\|\sigma_s\| \stackrel{def}{=} \|\sigma_s\|_{G(\psi) \rightarrow G(\psi)}.$$

It remains to prove (6). The upper bound is obtained as follows.

Let $f : f \in G(\psi)$, $f \neq 0$. We have

$$|\sigma_s f|_p^p = \int_0^\infty |f(x/s)|^p dx = s \int_0^\infty |f(y)|^p dy.$$

It follows from

$$\begin{aligned} |\sigma_s f|_p &= s^{1/p} |f|_p \leq \max(s^{1/a}, s^{1/b}) |f|_p \\ &\leq \max(s^{1/a}, s^{1/b}) \psi(p) \|f\|_{G(\psi)}; \end{aligned}$$

that

$$\|\sigma_s\| \leq \max(s^{1/a}, s^{1/b}).$$

For the lower bound, let $g(\cdot)$ be a representation of $\psi : |g|_p = \psi(p)$, $p \in (a, b)$; then $\|g\|_{G(\psi)} = 1$ and

$$\begin{aligned} \|\sigma_s\| &\geq \|\sigma_s g\|_{G(\psi)} = \sup_{p \in (a, b)} [|\sigma_s g|_p / \psi(p)] \\ &= \sup_{p \in (a, b)} [s^{1/p} |g|_p / \psi(p)] = \sup_{p \in (a, b)} s^{1/p} = \max(s^{1/a}, s^{1/b}). \end{aligned}$$

The proof is complete. \square

The proofs of our theorems 3, 4 and 6 go along similar lines and are strongly based on definitions and preliminary matter given above.

Proof of Theorem 3. Let $f \in G(\psi_1; a_1, b_1)$ and $g \in G(\psi_2; a_2, b_2)$. By definition of these spaces, this means

$$\begin{aligned} |f|_p &\leq \psi_1(p) \cdot \|f\|_{G(\psi_1)}, \quad p \in (a_1, b_1); \\ |g|_q &\leq \psi_2(q) \cdot \|g\|_{G(\psi_2)}, \quad q \in (a_2, b_2). \end{aligned}$$

It follows from Hölder's inequality that for $r \in (A_1, A_2)$ and $p, q > 1$, $1/p + 1/q = 1$,

$$|f * g|_r \leq |f|_{pr} |g|_{qr} \leq \psi_1(pr) \psi_2(qr) \|f\|_{G(\psi_1)} \|g\|_{G(\psi_2)}.$$

Minimizing the right-side over p and q provided $p, q > 1$ and $1/p + 1/q = 1$, we obtain the desired assertion. \square

Proof of Theorem 4. Here we will use the known Sobolev inequality in the L_p spaces (see, e.g., [9, Part 2, Ch.11, Sect.4] or for newer and more extended versions [18]) and [16]) which can be rewritten as

$$|Bu|_q \leq C_1(X, Y) q^{1-1/n} \|\text{grad } u\|_p, \quad p = qn/(q + m).$$

Let $|\text{grad } u| \in G(\psi; a, b)$ with $b < n$ (the case $b = n$ can be treated analogously); then we have for the values $q \in (A_2, B_2)$

$$\begin{aligned} |Bu|_q &\leq C_2(X, Y) \|\text{grad } u\|_{G(\psi)} q^{1-1/n} \psi(qn/(q + m)) \\ &\leq C_2(X, Y) \|\text{grad } u\|_{G(\psi)} \nu(q); \end{aligned}$$

and

$$\|Bu\|_{G(\nu)} \leq C_2(X, Y) \|\text{grad } u\|_{G(\psi)},$$

This completes the proof. \square

Proof of Theorem 6. Assume that $f \in G(\psi_1; a_1, b_1)$ and $g \in G(\psi_2; a_2, b_2)$. It follows from the definition of these spaces that for $p \in (a_1, b_1)$ and $q \in (a_2, b_2)$

$$|f|_p \leq \|f\|_{G(\psi_1)} \psi_1(p), \quad |g|_q \leq \|g\|_{G(\psi_2)} \psi_2(q).$$

Using the classical Young inequality ([2]), we obtain

$$|f * g|_r \leq C(p, q) |f|_p |g|_q, \quad C(p, q) \leq 1, \quad 1 + 1/r = 1/p + 1/q,$$

where for $n = \dim X$, $s = p/(p-1)$, $t = q/(q-1)$, and $z = r/(r-1)$ the constant $C(p, q)$ is given in [2] in the case $X = R^n$ in explicit form

$$C(p, q) = \left[p^{1/p} s^{-1/s} q^{1/q} t^{-1/t} r^{1/r} z^{-1/z} \right]^{n/2}.$$

This yields

$$|f * g|_r \leq \psi_1(p) \psi_2(q) \|f\|_{G(\psi_1)} \|g\|_{G(\psi_2)}.$$

The proof can now be completed as above, by minimizing over p and q , where $p, q > 1$ and $1/p + 1/q = 1 + 1/r$. \square

Proof of Theorem 5. We introduce the Sobolev-Grand Lebesgue spaces $W_1(\psi)$ with the (finite) norm of a function $u = u(x)$, defined on X ,

$$\|u\|_{W_1(\psi)} = \|\text{grad } u\|_{G(\psi)} + \|u\|_{G(\psi)}.$$

By this, the classical Sobolev embedding operator $S : W_1(\psi) \rightarrow G(\nu)$, $Su = Bu$, is not compact, unlike in the case of classical L_p spaces.

This fact follows from the assertion that for the considered function $u = u(|x|)$ the family of "small" shifts $\{T_\varepsilon u(|x|) = u(\varepsilon + |x|)\}$, with $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$, has a positive distance in both $G(\psi)$ and $W_1(\psi)$ spaces: there exists $C > 0$ such that

$$\|T_\varepsilon u - T_\delta u\|_{G(\psi)} \geq C$$

and

$$\|T_\varepsilon u - T_\delta u\|_{W_1(\psi)} \geq C,$$

where $\varepsilon, \delta \in (0, \varepsilon_0)$, $\varepsilon \neq \delta$.

In order to prove the first assertion, (the second may be proved analogously), we introduce certain subspaces of $G(\psi; a, b)$ space. Let us denote

$$GA(\psi) = GA(\psi; a, b) = \{f : \lim_{\delta \rightarrow 0+} \sup_{A: \mu(A) \leq \delta} \|f I(A)\|_{G(\psi)} = 0\};$$

and $GB(\psi) = GB(\psi; a, b)$ as the set of all f such that for all $\varepsilon > 0$ there exist $B \in (0, \infty)$ and $A \in \Sigma$, with $\mu(A) \leq B$, and there exists $g : X \rightarrow \mathbb{R}$ such that $g(x) = g(x)I(A)$ and $\sup_x |g(x)| < B$ and $\|f - g\| < \varepsilon$. Let also

$$G^0(\psi; a, b) = G^0(\psi) = \{f : \lim_{\psi(p) \rightarrow \infty} |f|_p / \psi(p) = 0\}.$$

The spaces $GA(\psi), GB(\psi), G^0(\psi)$ are closed subspaces of $G(\psi)$. We assume here that there is $h : X \rightarrow \mathbb{R}$, $|h|_p \asymp \psi(p)$, $p \in (a, b)$.

It follows from the theory of r.i. spaces ([1, Ch.1, p.p.22 - 28]) that if $\psi \in \Psi$, then

$$GA(\psi) = GB(\psi) = G^0(\psi) \neq G(\psi).$$

Without loss of generality we can assume $X = (0, 2\pi]$, $\sigma = 0$, and define also $x \pm y = x \pm y \pmod{2\pi}$ for $x, y \in X$.

We take a function $u = u(x) \in G(\psi) \setminus G^0(\psi)$, $x \in X$. Then (see [1, Ch.3, p.p.192 - 198])

$$\inf_{\varepsilon \neq \delta} \|T_\varepsilon u - T_\delta u\|_{G(\psi)} = \inf_{\varepsilon \neq \delta} \|T_{\varepsilon-\delta} u - u\|_{G(\psi)} > 0.$$

The proof is complete. □

4. SHARPNESS

We will discuss either the sharpness or lack of that for obtained results. Since the sharpness of Theorems 2, 3 and 4 is obtained readily, we did not postpone it and gave immediately after the formulations.

Let us demonstrate (briefly) the sharpness of Theorem 6 by considering only the case $n = 1$ and $\sigma = 0$. Let both $\gamma_1, \gamma_2 \geq 0$, and define

$$\begin{aligned} f(x) &= I(0 < x < 1) x^{-1/b_1} |\log x|^{\gamma_1}, \quad b_1, b_2 > 1, \\ g(x) &= I(0 < x < 1) x^{-1/b_2} |\log x|^{\gamma_2}, \quad h(t) = (f * g)(t). \end{aligned}$$

It suffices to consider the case $t \in (0, 1/4)$, since on $[1/4, 2]$ the function $h = h(t)$ is bounded and for $t \in (-\infty, 0) \cup (2, \infty)$ this function vanishes.

For $t \rightarrow 0+$ we get

$$\begin{aligned}
h(t) &= \int_0^t x^{-1/b_1} |\log x|^{\gamma_1} (t-x)^{-1/b_2} |\log(t-x)|^{\gamma_2} dx \\
&= t^{1-1/b_1-1/b_2} \int_0^1 y^{-1/b_1} (1-y)^{-1/b_2} |\log t + \log y|^{\gamma_1} |\log t + \log(1-y)|^{\gamma_2} dy \\
&\sim t^{1-1/b_1-1/b_2} |\log t|^{\gamma_1+\gamma_2} \int_0^1 y^{-1/b_1} (1-y)^{-1/b_2} dy \\
&= B(1-1/b_1, 1-1/b_2) t^{1-1/b_1-1/b_2} |\log t|^{\gamma_1+\gamma_2};
\end{aligned}$$

here $B(\cdot, \cdot)$ stands for the beta-function. Taking then the $|\cdot|_p$ norm, we obtain

$$|f * g|_p \sim C(B_3 - p)^{-\gamma_1-\gamma_2-1/b_1-1/b_2+1}, \quad p \in [1, B_3),$$

The situation is more complicated with (4) and (5). Roughly speaking, we can present examples when the inequalities are achieved but for the same examples the actual bounds are better. In other words, the sharpness of these inequalities is an open problem.

We first analyze (4). Let $\psi(p) = (p-a)^{-\alpha} (b-p)^{-\beta}$, $\sigma = 0$, $1 \leq a < b < n$. Consider the function $u = u(|x|)$, with $X = R^n$ and $Y = R^m$, for which $|\text{grad } u| \in G(a, b; \alpha, \beta)$, i.e., for $p \rightarrow a+0$ and $p \rightarrow b-0$ we have $|\text{grad } u|_p \sim (p-a)^{-\alpha} (b-p)^{-\beta}$, then it follows from the inequality (4) that

$$|Bu|_p \leq C(p-A_2)^{-\alpha} (B_2-p)^{-\beta}, \quad p \in (A_2, B_2).$$

However, in the considered case for the same range of p

$$|Bu|_p \sim C(p-A_2)^{-\alpha+1/n} (B_2-p)^{-\beta+1/n}.$$

In the same way, analogous examples may be constructed in the cases when either $b_1 = \infty$ or $b_2 = \infty$.

Therefore, the bounds A_2 and B_2 are, in general, exact, but between the *exponents* obtained $-\alpha$ and $-\beta$ on one side and $-\alpha+1/n$ and $-\beta+1/n$ in the example on the other side there is the $1/n$ "gap".

But since the dimension n may be sufficiently great, we can conclude that in general case the assertion (3) of theorem 4 is exact.

We give a similar example for (5). Let $X = R$, $\sigma = 0$, $\gamma_1, \gamma_2 \geq 0$, $1 \leq a_1, a_2 < b_1, b_2 < \infty$, and $1/b_1 + 1/b_1 > 1$. Considering the functions

$$\begin{aligned}
f(x) &= I(0 < |x| < 1) |x|^{-1/b_1} |\log |x||^{\gamma_1}, \\
g(x) &= I(0 < |x| < 1) |x|^{-1/b_2} |\log |x||^{\gamma_2},
\end{aligned}$$

and $h = f * g$, we then have

$$f \in G(1, b_1; 0, \gamma_1 + 1/b_1), \quad g \in G(1, b_2; 0, \gamma_2 + 1/b_2).$$

It follows from (5) that for $p \in [1, B_3)$

$$|h|_p \leq C(B_3 - p)^{-\gamma_1 - \gamma_2 - 1/b_1 - 1/b_2}.$$

In fact

$$|h|_p \sim C(B_3 - p)^{-\gamma_1 - \gamma_2 - 1/b_1 - 1/b_2 + 1}.$$

Analogously, if $h = f * g$, where

$$\begin{aligned} f(x) &= I(|x| > 1) |x|^{-1/a_1} |\log |x||^{\gamma_1}, \\ g(x) &= I(|x| > 1) |x|^{-1/a_2} |\log |x||^{\gamma_2}, \end{aligned}$$

then

$$f \in G(a_1, b_1; \gamma_1 + 1/a_1, 0), \quad g \in G(a_2, b_2; \gamma_2 + 1/b_2, 0).$$

It follows from (5) that

$$|h|_p \leq C(p - A_3)^{-\gamma_1 - \gamma_2 - 1/a_1 - 1/a_2},$$

but in fact

$$|h|_p \sim C(p - A_3)^{-\gamma_1 - \gamma_2 - 1/a_1 - 1/a_2 + 1}.$$

Therefore, like above the *bounds* A_3 and B_3 are in this case exact, but between the *exponents* there is a 1 "gap".

Note that in the considered case the "gap" does not tends to zero as $n \rightarrow \infty$, opposite to the previous (convolution) case.

Finding sharp estimates in these two cases is an interesting open problem.

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